The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2018 Fryer Contest

Thursday, April 12, 2018
(in North America and South America)

Friday, April 13, 2018
(outside of North America and South America)

Solutions

1. (a) On Monday, Shane bought 4 boxes of cherries at a cost of $\$2.00 \times 4 = \$8.00$.
He also bought 3 boxes of plums at a cost of $\$3.00 \times 3 = \$9.00$, and 2 boxes of blueberries at a cost of $\$4.50 \times 2 = \$9.00$.
In total, Shane paid $\$8.00 + \$9.00 + \$9.00 = \$26.00$.

(b) On Wednesday, Shane bought 2 boxes of plums at at cost of $\$3.00 \times 2 = \$6.00$.
Since Shane spent \$22.00 in total, he spent $\$22.00 - \$6.00 = \$16.00$ on boxes of cherries.
The price of each box of cherries is \$2.00, and so Shane bought $\$16.00 \div \$2.00 = 8$ boxes of cherries.

(c) *Solution 1*
On Saturday, Shane gave the cashier \$100.00 and received \$14.50 in change, and so Shane spent $\$100.00 - \$14.50 = \$85.50$.
Of the \$85.50, Shane spent $\$4.50 \times 3 = \$13.50$ on 3 boxes of blueberries.
Therefore, Shane spent $\$85.50 - \$13.50 = \$72.00$ on boxes of plums and boxes of cherries.
If Shane bought c boxes of cherries, then he bought twice as many boxes or $2c$ boxes of plums.
The cost of c boxes of cherries is $\$2.00 \times c = \$2c$.
The cost of $2c$ boxes of plums is $\$3.00 \times 2c = \$6c$.
Shane spent a total of $\$2c + \$6c = \$8c$ on boxes of plums and boxes of cherries, and so $8c = 72$ or $c = 9$.
Therefore, Shane bought 9 boxes of cherries.

Solution 2
As in Solution 1, we begin by determining that Shane spent \$72.00 on boxes of plums and boxes of cherries.
For every 1 box of cherries that Shane bought, he purchased 2 boxes of plums.
The cost of 1 box of cherries and 2 boxes of plums is $\$2.00 + 2 \times \$3.00 = \$8.00$ and $\$72.00 \div \$8.00 = 9$, so Shane bought 9 boxes of cherries (and $2 \times 9 = 18$ boxes of plums).

Note: In both solutions, we may check that the cost of 9 boxes of cherries, 18 boxes of plums, and 3 boxes of blueberries is $9 \times \$2.00 + 18 \times \$3.00 + 3 \times \$4.50 = \$85.50$, and so the correct change from \$100.00 is \$14.50, as expected.

2. (a) In the first diagram (Paul's Path), $\triangle ABM$ is right-angled at B.
By the Pythagorean Theorem, $MA^2 = AB^2 + BM^2$ or $MA^2 = 105^2 + 100^2 = 21\,025$ or $MA = \sqrt{21\,025} = 145$ m (since $MA > 0$).

(b) In the second diagram (Tyler's Path), $AD = BC = 200$ m and $DC = AB = 105$ m (since $ABCD$ is a rectangle).
Thus $PD = AD - AP = 200 - 140 = 60$ m.
Also, $DQ = DC - QC = 105 - 60 = 45$ m.
Since $\triangle PDQ$ is right-angled at D, using the Pythagorean Theorem, we get $PQ^2 = PD^2 + DQ^2$ or $PQ^2 = 60^2 + 45^2 = 5625$ or $PQ = \sqrt{5625} = 75$ m (since $PQ > 0$).
The total distance that Tyler runs is

$$AP + PQ + QC + CB + BA = 140 + 75 + 60 + 200 + 105 = 580 \text{ m.}$$

(c) The total distance that Paul runs is

$$AD + DC + CM + MA = 200 + 105 + (200 - 100) + 145 = 550 \text{ m.}$$

Tyler runs at a speed of 145 m/min, and so it takes Tyler $580 \div 145 = 4$ min to finish his path.

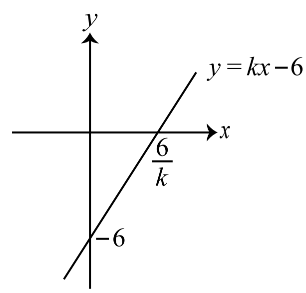
Paul begins at the same time as Tyler and finishes his path 1 minute after Tyler, and so Paul takes $4 + 1 = 5$ min to finish his path.
In this time, Paul runs a total distance of 550 m and so Paul's speed is $550 \div 5 = 110$ m/min.

3. (a) We determine the x-intercept by letting $y = 0$ in the equation $y = 2x - 6$ and solving for x.
Thus, $0 = 2x - 6$ and so $2x = 6$ or $x = 3$.
The x-intercept of the line with equation $y = 2x - 6$ is 3.

We determine the y-intercept by letting $x = 0$ in the equation $y = 2x - 6$ and solving for y.
Thus, $y = 2(0) - 6$ and so $y = -6$.
The y-intercept of the line with equation $y = 2x - 6$ is -6.

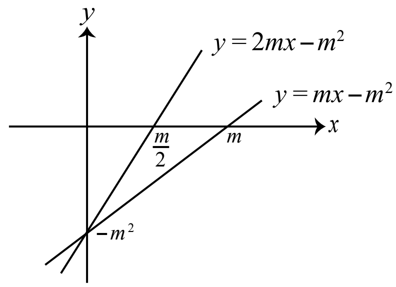
(b) Letting $y = 0$, we get $0 = kx - 6$ or $kx = 6$ and so $x = \frac{6}{k}$, where $k \neq 0$.
The line with equation $y = kx - 6$ has x-intercept $\frac{6}{k}$ $(k \neq 0)$.

(c) From part (b), the line with equation $y = kx - 6$ has x-intercept $\frac{6}{k}$.
Since $k > 0$, then $\frac{6}{k} > 0$ and so the line intersects the positive x-axis.
The y-intercept of the line with equation $y = kx - 6$ is -6.



The triangle formed by the line with equation $y = kx - 6$ $(k > 0)$, the positive x-axis, and the negative y-axis, has area $\frac{1}{2}\left(\frac{6}{k}\right)(6) = \frac{36}{2k} = \frac{18}{k}$ (the y-intercept is -6, and so the triangle has height 6).
Since the area of this triangle is 6, then $\frac{18}{k} = 6$ or $18 = 6k$ and so $k = 3$.

(d) The x-intercept of the line with equation $y = 2mx - m^2$ is determined by letting $y = 0$ and solving for x.
Thus, $0 = 2mx - m^2$ or $0 = m(2x - m)$ and since $m > 0$, then $2x = m$ or $x = \frac{m}{2}$.
The x-intercept of this line is $\frac{m}{2}$ $(m > 0)$.
The y-intercept of the line with equation $y = 2mx - m^2$ is $2m(0) - m^2 = -m^2$.
Similarly, the x-intercept of the line with equation $y = mx - m^2$ is given by $0 = mx - m^2$ or $0 = m(x - m)$ and since $m > 0$, then $x = m$.
The x-intercept of this line is m $(m > 0)$.
The y-intercept of the line with equation $y = mx - m^2$ is $m(0) - m^2 = -m^2$.
Thus, both lines have the same y-intercept.



To determine the area of the triangle formed by the positive x-axis, the line with equation $y = mx - m^2$, and the line with equation $y = 2mx - m^2$ $(m > 0)$, we may let the length of the base be the distance between the x-intercepts or $m - \frac{m}{2} = \frac{m}{2}$.
Then the height of this triangle is the perpendicular distance from the x-axis to the y-intercept, or m^2 (the y-intercept is $-m^2$, and so the triangle has height m^2, a positive number).
Therefore, the triangle has area $\frac{1}{2}\left(\frac{m}{2}\right)(m^2) = \frac{m^3}{4}$.
The area of this triangle is $\frac{54}{125}$ and so $\frac{m^3}{4} = \frac{54}{125}$ or $m^3 = \frac{216}{125}$ and so $m = \sqrt[3]{\frac{216}{125}} = \frac{6}{5}$.
(Note that $\left(\frac{6}{5}\right)^3 = \frac{216}{125}$.)
The only value of m for which the triangle has area $\frac{54}{125}$ is $m = \frac{6}{5}$.

4. (a) There are 2 choices (a 1 or a 2) for each of the 3 digits, and so there are $2^3 = 8$ 3-digit Bauman numbers.
These 3-digit Bauman numbers are: 111, 112, 121, 122, 211, 212, 221, 222.

(b) Every Bauman number having fewer than three blocks has exactly one block or it has exactly two blocks.

First we consider 10-digit Bauman numbers that have exactly one block.
If a 10-digit Bauman number has exactly 1 block, then the number is made up of 10 ones or it is made up of 10 twos.
Thus, the number of 10-digit Bauman numbers having exactly one block is 2.

Next we consider 10-digit Bauman numbers that have exactly two blocks.
If the Bauman number has exactly two blocks, then it has a block of 1s followed by a block of 2s or it has a block of 2s followed by a block of 1s.
To begin, assume that the block of 1s comes before the block of 2s.
The block of 1s could be 1 digit in length, 2 digits in length, and so on up to 9 digits in length (9 is the maximum length since a block of 2s must follow the block of 1s).
In each case, the remaining digits in the number are all 2s and so there are exactly 9 Bauman numbers of this form.
Similarly, there are 9 Bauman numbers with exactly two blocks whose first block is made up of 2s (from 1 to 9 of them) and whose remaining digits are 1s.

In total, there are $2+9+9 = 20$ Bauman numbers consisting of 10-digits and having fewer than three blocks.

(c) We first consider Bauman numbers that consist of exactly one block.
Any number of 2s will sum to an even number, and so there are no Bauman numbers consisting of a single block of 2s whose digit sum is 7.

There is 1 Bauman number consisting of a single block of seven 1s whose digit sum is 7.

Next, we consider Bauman numbers that consist of exactly two blocks.
In this case, the number must have at least one 2 (since there are two blocks), and at most three 2s, since the digit sum is 7.
If the Bauman number has a block consisting of a single 2, then the second block must consist of five 1s.
There are exactly 2 numbers having this form: 211 111 and 111 112.
If the Bauman number has a block consisting of two 2s, then the second block must consist of three 1s.
There are exactly 2 numbers having this form: 22 111 and 11 122.
If the Bauman number has a block consisting of three 2s, then the second block must consist of one 1.
There are exactly 2 numbers having this form: 2221 and 1222.
There are 6 Bauman numbers consisting of exactly two blocks and whose digit sum is 7.

Finally, we consider Bauman numbers that consist of exactly three blocks.
As before, the number must have at least one 2 (since there are three blocks) and at most three 2s, since the digit sum is 7.
Bauman numbers of this form must consist of:

(i) one block of one 2 and two blocks of 1s (five 1s in total), or
(ii) one block of two 2s and two blocks of 1s (three 1s in total), or
(iii) one block of one 1 and two blocks of 2s (three 2s in total), or
(iv) one block of three 1s and two blocks of 2s (two 2s in total)

We note that it is not possible for Bauman numbers of this form to consist of:

- one block of three or more 2s and two blocks of 1s since the digit sum would be greater than 7;
- one block of two 1s and two blocks of 2s since the digit sum would be even;
- one block of four or more 1s and two blocks of 2s since the digit sum would be greater than 7.

In the table below, we list the Bauman numbers having exactly 3 blocks and whose digit sum is 7.
Each row of the table corresponds to one of the four possible cases outlined above.

Case	Bauman numbers
(i)	121 111, 112 111, 111 211, 111 121
(ii)	12 211, 11 221
(iii)	2122, 2212
(iv)	21 112

There are 9 Bauman numbers consisting of exactly three blocks and whose digit sum is 7.

The number of Bauman numbers that consist of at most three blocks and have the property that the sum of the digits is 7 is $1 + 6 + 9 = 16$.

(d) We introduce the notation $\boxed{2}$ to represent a block of exactly 2018 2s, and X_n to represent a string of n digits, in which each digit can be either a 1 or a 2.
We are asked to determine the number of 4037-digit Bauman numbers that include at least one $\boxed{2}$.
We consider the following three cases.

(i) The Bauman number begins with a $\boxed{2}$. That is, the first 2018 digits are 2s and the 2019^{th} digit is a 1. We note that in this case the 2019^{th} digit must be a 1, otherwise the block at the beginning of this number would have at least 2019 2s and thus would not begin with a $\boxed{2}$.

(ii) The Bauman number ends with a $\boxed{2}$. That is, the last 2018 digits are 2s and the digit preceding these digits (which is again the 2019^{th} digit in the number) is a 1.
It is worth noting here that both (i) and (ii) can occur simultaneously.

(iii) The Bauman number contains a $\boxed{2}$, however the $\boxed{2}$ does not occur at the beginning of the number and it does not occur at the end of the number. In this case, the Bauman number contains the 2020 digits $1\boxed{2}1$, in this order.

We note that every Bauman number that contains at least one $\boxed{2}$ must satisfy the conditions in at least one of these three cases above.
Next, we count the number of 4037-digit Bauman numbers that are in each of these three cases.

<u>Case (i)</u>
The first 2019 digits of the number are $\boxed{2}1$, and so there are $4037 - 2019 = 2018$ digits remaining, each of which can be either a 1 or a 2.
In this case, the Bauman numbers are of the form $\boxed{2}1X_{2018}$.
There are two choices for each of the remaining 2018 digits (each can be a 1 or a 2), and so there are 2^{2018} Bauman numbers of this form.

<u>Case (ii)</u>
Similarly, the last 2019 digits of the number are $1\boxed{2}$, and so there are $4037 - 2019 = 2018$ digits remaining, each of which can be a 1 or a 2.
In this case, the Bauman numbers are of the form $X_{2018}1\boxed{2}$.
There are two choices for each of the remaining 2018 digits (each can be a 1 or a 2), and so there are 2^{2018} Bauman numbers of this form.
As was noted earlier, there is exactly 1 number which satisfies the conditions of both Case (i) and Case (ii).
The Bauman number $\boxed{2}1\boxed{2}$ has 4037 digits, and begins and ends with a $\boxed{2}$.
That is, we have counted this number twice, once in Case (i) and again in Case (ii).
Therefore, the total number of 4037-digit Bauman numbers that satisfy the conditions of Case (i) or Case (ii) is $2^{2018} + 2^{2018} - 1$.

<u>Case (iii)</u>
We first note that every Bauman number satisfying the conditions of Case (iii) must be different than every Bauman number satisfying the conditions of Case (i) or Case (ii).
Begin by assuming that the first 2020 digits of the Bauman number are $1\boxed{2}1$.
In this case, there are $4037 - 2020 = 2017$ digits remaining, each of which can be a 1 or a 2.
These Bauman numbers are of the form $1\boxed{2}1X_{2017}$.
Since only 2017 digits remain to be chosen, it is not possible to have a second $\boxed{2}$ since each $\boxed{2}$ contains 2018 digits.
There are two choices for each of the remaining 2017 digits, and so there are 2^{2017} Bauman numbers of this form.
We may move the string of 2020 digits $1\boxed{2}1$ to the right 1 digit to give a number having the new form $X_1 1\boxed{2}1X_{2016}$.
Again, there are 2^{2017} ways to replace the 2017 digits.

Continuing to move the string of 2020 digits 1 digit to the right, we get numbers of the form $X_2 1\boxed{2}1 X_{2015}$, $X_3 1\boxed{2}1 X_{2014}$, $X_4 1\boxed{2}1 X_{2013}$, and so on until the string $1\boxed{2}1$ is at the end of the number and we obtain a number of the form $X_{2017} 1\boxed{2}1$.
In each of these, there are two choices for each of the remaining 2017 digits, and so there are 2^{2017} Bauman numbers of this form.
That is, for each Bauman number having one of the forms:

$$1\boxed{2}1X_{2017}, X_1 1\boxed{2}1X_{2016}, X_2 1\boxed{2}1X_{2015}, X_3 1\boxed{2}1X_{2014}, \ldots, X_{2016} 1\boxed{2}1X_1, X_{2017} 1\boxed{2}1,$$

there are 2^{2017} ways to fill in the remaining digits.
Since there are 2018 of these forms, each having 2^{2017} ways to fill in the remaining digits, then there are $2018 \cdot 2^{2017}$ Bauman numbers satisfying the conditions of Case (iii).

The number of 4037-digit Bauman numbers that include at least one block of exactly 2018 2s is

$$\begin{aligned} 2^{2018} + 2^{2018} - 1 + 2018 \cdot 2^{2017} &= 2 \cdot 2^{2018} - 1 + 1009 \cdot 2 \cdot 2^{2017} \\ &= 2 \cdot 2^{2018} - 1 + 1009 \cdot 2^{2018} \\ &= 1011 \cdot 2^{2018} - 1. \end{aligned}$$