The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Euclid Contest

Tuesday, April 12, 2016
(in North America and South America)

Wednesday, April 13, 2016
(outside of North America and South America)

UNIVERSITY OF
WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express calculations and answers as exact numbers such as $\pi + 1$ and $\sqrt{2}$, etc., rather than as $4.14\ldots$ or $1.41\ldots$, except where otherwise indicated.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:

1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked , place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked , provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

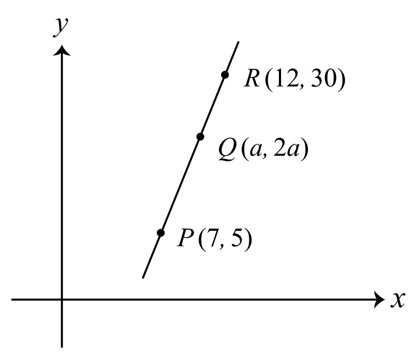
A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

1. (a) What is the average of the integers $5, 15, 25, 35, 45, 55$?

   (b) If $x^2 = 2016$, what is the value of $(x + 2)(x - 2)$?

   (c) In the diagram, points $P(7, 5)$, $Q(a, 2a)$, and $R(12, 30)$ lie on a straight line. Determine the value of a.

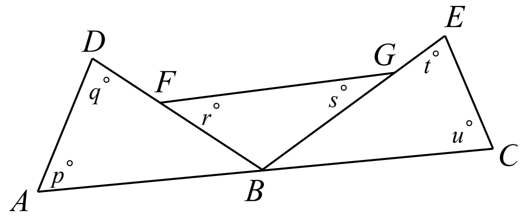


2. (a) What are all values of n for which $\dfrac{n}{9} = \dfrac{25}{n}$?

   (b) What are all values of x for which $(x - 3)(x - 2) = 6$?

   (c) At Willard's Grocery Store, the cost of 2 apples is the same as the cost of 3 bananas. Ross buys 6 apples and 12 bananas for a total cost of \$6.30. Determine the cost of 1 apple.

3. (a) In the diagram, point B is on AC, point F is on DB, and point G is on EB.



What is the value of $p + q + r + s + t + u$?

(b) Let n be the integer equal to $10^{20} - 20$. What is the sum of the digits of n?

(c) A parabola intersects the x-axis at $P(2, 0)$ and $Q(8, 0)$. The vertex of the parabola is at V, which is below the x-axis. If the area of $\triangle VPQ$ is 12, determine the coordinates of V.

4. (a) Determine all angles θ with $0^\circ \leq \theta \leq 180^\circ$ and $\sin^2\theta + 2\cos^2\theta = \frac{7}{4}$.

(b) The sum of the radii of two circles is 10 cm. The circumference of the larger circle is 3 cm greater than the circumference of the smaller circle. Determine the difference between the area of the larger circle and the area of the smaller circle.

5. (a) Charlotte's Convenience Centre buys a calculator for $\$p$ (where $p > 0$), raises its price by $n\%$, then reduces this new price by 20%. If the final price is 20% higher than $\$p$, what is the value of n?

(b) A function f is defined so that if n is an odd integer, then $f(n) = n - 1$ and if n is an even integer, then $f(n) = n^2 - 1$. For example, if $n = 15$, then $f(n) = 14$ and if $n = -6$, then $f(n) = 35$, since 15 is an odd integer and -6 is an even integer. Determine all integers n for which $f(f(n)) = 3$.

6. (a) What is the smallest positive integer x for which $\dfrac{1}{32} = \dfrac{x}{10^y}$ for some positive integer y?

(b) Determine all possible values for the area of a right-angled triangle with one side length equal to 60 and with the property that its side lengths form an arithmetic sequence.

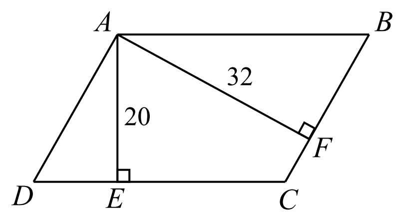
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9 are the first four terms of an arithmetic sequence.)

7. (a) Amrita and Zhang cross a lake in a straight line with the help of a one-seat kayak. Each can paddle the kayak at 7 km/h and swim at 2 km/h. They start from the same point at the same time with Amrita paddling and Zhang swimming. After a while, Amrita stops the kayak and immediately starts swimming. Upon reaching the kayak (which has not moved since Amrita started swimming), Zhang gets in and immediately starts paddling. They arrive on the far side of the lake at the same time, 90 minutes after they began. Determine the amount of time during these 90 minutes that the kayak was not being paddled.

(b) Determine all pairs (x, y) of real numbers that satisfy the system of equations

$$\begin{aligned} x\left(\tfrac{1}{2} + y - 2x^2\right) &= 0 \\ y\left(\tfrac{5}{2} + x - y\right) &= 0 \end{aligned}$$

8. (a) In the diagram, $ABCD$ is a parallelogram. Point E is on DC with AE perpendicular to DC, and point F is on CB with AF perpendicular to CB. If $AE = 20$, $AF = 32$, and $\cos(\angle EAF) = \frac{1}{3}$, determine the exact value of the area of quadrilateral $AECF$.

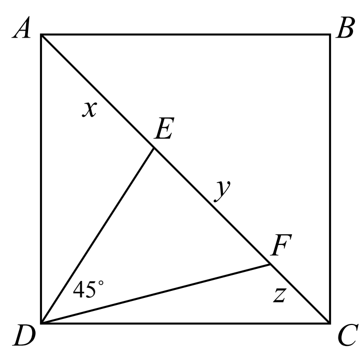


(b) Determine all real numbers $x > 0$ for which

$$\log_4 x - \log_x 16 = \tfrac{7}{6} - \log_x 8$$

9. (a) The string $AAABBBAABB$ is a string of ten letters, each of which is A or B, that does not include the consecutive letters $ABBA$.
The string $AAABBAAABB$ is a string of ten letters, each of which is A or B, that does include the consecutive letters $ABBA$.
Determine, with justification, the total number of strings of ten letters, each of which is A or B, that do not include the consecutive letters $ABBA$.

(b) In the diagram, $ABCD$ is a square. Points E and F are chosen on AC so that $\angle EDF = 45^\circ$. If $AE = x$, $EF = y$, and $FC = z$, prove that $y^2 = x^2 + z^2$.



10. Let k be a positive integer with $k \geq 2$. Two bags each contain k balls, labelled with the positive integers from 1 to k. André removes one ball from each bag. (In each bag, each ball is equally likely to be chosen.) Define $P(k)$ to be the probability that the product of the numbers on the two balls that he chooses is divisible by k.

(a) Calculate $P(10)$.

(b) Determine, with justification, a polynomial $f(n)$ for which

- $P(n) \geq \dfrac{f(n)}{n^2}$ for all positive integers n with $n \geq 2$, and
- $P(n) = \dfrac{f(n)}{n^2}$ for infinitely many positive integers n with $n \geq 2$.

(A *polynomial* $f(x)$ is an algebraic expression of the form $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ for some integer $m \geq 0$ and for some real numbers $a_m, a_{m-1}, \ldots, a_1, a_0$.)

(c) Prove there exists a positive integer m for which $P(m) > \dfrac{2016}{m}$.

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For students...

Thank you for writing the 2016 Euclid Contest! Each year, more than 220 000 students from more than 60 countries register to write the CEMC's Contests.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2016 Canadian Senior Mathematics Contest, which will be written in November 2016.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

For teachers...

Visit our website `cemc.uwaterloo.ca` to

- Obtain information about our 2016/2017 contests
- Look at our free online courseware for high school students
- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results